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# VIBRATION OF A STAMP ON A TWO-LAYER BASE

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The frictionless vibration of a circular stamp on an elastic two-layer base is considered. The boundary value problem reduces to an integral equation of the first kind. Its unique solvability is proved in some class of functions and an approximate method of solution is proposed, based on the factorization of the functions. Programs are constructed which realize the method on a digital computer, and results are presented of a numerical analysis of the solution.

**1.** By known methods taking account of radiation conditions, a boundary value problem is reduced to the solution of an integral equation of the following kind ($q(\rho)$ is the amplitude value of the dimensionless contact stress under the stamp, and $f(r)$ is the amplitude value of the stamp vibrations at the point $r$):

$$K_q^\circ \equiv \int_0^a k(r, \rho)\, q(\rho)\, \rho\, d\rho = 8\pi f(r) \tag{1.1}$$

$$k(r, \rho) = \int_\Gamma K(u)\, J_0(ur)\, J_0(u\rho)\, u\, du$$

$$K(u) = H_1(u) + \frac{H(u)\, l_1^2(u) - 2L(u)\, l_1(u)\, h_1 u + M(u)\, h_1^2(u)}{L^2(u) - M(u)\, H(u)} \tag{1.2}$$

Here

$$L(u) = L_1(u) - L_2(u), \quad M(u) = M_1(u) + M_2(u)$$

$$H(u) = H_1(u) + H_2(u), \quad L_i(u) = L_i^+(u) + L_i^-(u)$$

$$M_i(u) = M_i^+(u) + M_i^-(u), \quad H_i^+(u) + H_i^-(u)$$

$$l_1(u) = L_1^-(u) - L_1^+(u), \quad h_1(u) = H_1^-(u) - H_1^+(u)$$

$$L_1^+(u) = \frac{\gamma_1 \operatorname{ch} \sigma_1 \operatorname{sh} \sigma_2 - \sigma_1\sigma_2 \operatorname{ch} \sigma_2 \operatorname{sh} \sigma_1}{\mu_1 \Delta_1^+(u)}$$

$$M_1^+(u) = -\frac{\theta_2^2 \sigma_2 \operatorname{ch} \sigma_1 \operatorname{ch} \sigma_2}{2\mu_1 u^2 \Delta_1^+(u)}, \quad H_1^+(u) = -\frac{\theta_2^2 \sigma_1 \operatorname{sh} \sigma_1 \operatorname{sh} \sigma_2}{2\mu_1 \Delta_1^+(u)}$$

$$\Delta_1^+(u) = \gamma_1^2 \operatorname{ch} \sigma_1 \operatorname{sh} \sigma_2 - u^2 \sigma_1 \sigma_2 \operatorname{sh} \sigma_1 \operatorname{ch} \sigma_2$$

$$L_2^+(u) = L_2^-(u) = \frac{\gamma_1'^2 - \sigma_1' \sigma_2'}{\mu_2 \Delta_2(u)}$$

$$M_2^+(u) = M_2^-(u) = -\frac{\theta_2'^2 \sigma_2'}{2\mu_2 u^2 \Delta_2(u)}$$

$$H_2^+(u) = H_2^-(u) = -\frac{\theta_2'^2 \sigma_1'}{2\mu_2 \Delta_2(u)}$$

$$\Delta_2(u) = \gamma_1'^2 - u^2 \sigma_1' \sigma_2', \quad \gamma_1 = u^2 - {}^1/_2 \theta_1^2$$

$$\theta_1^2 = (1 + \nu_1)(1 - 2\nu_1)\, \varepsilon^2 \varkappa_2^2 / (1 - \nu_1)$$

$$\theta_2^2 = 2(1 + \nu_1)\, \varepsilon^2 \varkappa_2^2, \quad \sigma_i^2 = u^2 - \theta_i^2$$

$$\theta_1'^2 = (1 + \nu_2)(1 - 2\nu_2)\varkappa_2^2 / (1 - \nu_2), \quad \gamma_1' = u^2 - {}^1/_2\theta_1'^2$$
$$\theta_2'^2 = 2(1 + \nu_2)\varkappa_2^2, \quad \sigma_i' = u^2 - \theta_i'^2, \quad \varkappa_2^2 = \rho_2\omega^2 h^2 / E_2$$
$$\varepsilon = \rho_1 E_2 / (\rho_2 E_1), \quad \mu_i = E_i / [2(1 + \nu_i)], \; a = A / h, \quad r = R / h,$$
$$u = Uh$$

The functions $L_1^-(u)$, $M_1^-(u)$, $H_1^-(u)$, $\Delta_1^-(u)$ are obtained from $L_1^+(u)$, $M_1^+(u)$, $H_1^+(u)$, $\Delta_1^+(u)$ by replacing the sinh by cosh, cosh by sinh in all the expressions; $\nu_1$, $\nu_2$ are the Poisson's ratios, $\rho_1$, $\rho_2$ are densities, $E_1$, $E_2$ are the Young's moduli for the material of the layer and the material of the half-space, respectively, $A$ is the dimensional radius of the stamp, $h$ is the dimensional layer thickness, and $\omega$ is the frequency of stamp vibration (the time dependence is described by the function $e^{-i\omega t}$).

We note the following general properties of the function $K(u)$. This is an even function, analytic in a complex plane with slits in the first and third quadrants which connect the points $u = \pm\theta_1'$ and $u = \pm\theta_2'$ with a point at infinity. Moreover

$$K(u) = \frac{c^2}{|u|}[1 + O(u^{-1})],$$
$$|u| \to \infty; \quad c^2 = \frac{2\theta_2^2}{\mu_1(\theta_2^2 - \theta_1^2)}$$

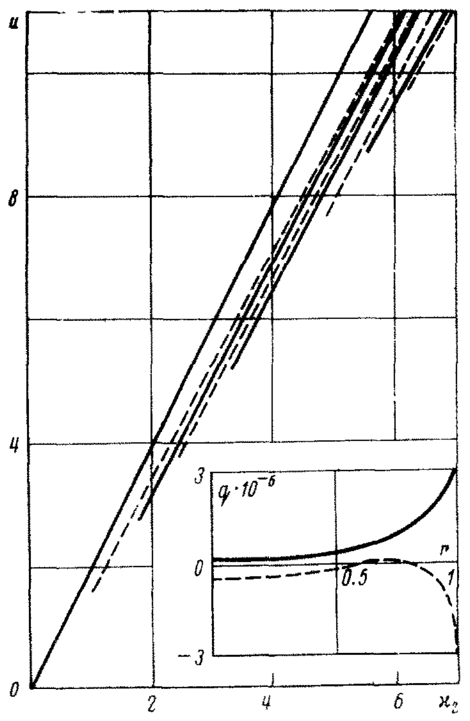


Fig. 1

In addition to the branch points, the function $K(u)$ can also have zeros and poles on the real axis. Their distribution depends essentially on both the elastic and geometric parameters of the problem and on the frequency of system vibration.

The complex form of the function $K(u)$ permits studying the distribution of the real zeros and poles in each specific case only by using a digital computer. Curves of the zeros (dashes) and poles (solid lines) of the function $K(u)$ as a function of the reduced frequency $\varkappa_2$ for $\varepsilon^2 = 1.25$, $\nu_1 = 0.27$, $\nu_2 = 0.26$, $E_1 = 1.2\cdot 10^6$ and $E_2 = 1.5\cdot 10^6$ are presented in Fig. 1.

The contour $\Gamma$ in the representation of the kernel of the integral equation (1. 1) lies in the right half-plane and coincides everywhere with the real axis, with the exception of domains containing the real poles, zeros, and branch points which are bypassed from below.

**2.** The method used later consists in replacing the initial integral equation by some approximation which is solved sufficiently easily. Use of this method is possible only for unique solvability of the initial equation [1]. In this connection, let us prove the unique solvability of (1. 1) in the case under consideration.

The following uniqueness theorem is valid for integral equations of the form (1. 1).

Let an even function $K(u)$ with the analyticity properties listed above (the presence of real branch points and single poles) and with the behavior (1.3) be such that it has domains $[A_{2k}, A_{2k+1}]$ on the real axis where it is real, and domains $[B_{2k}, B_{2s+1}]$ where it is complex. Let the zeros and poles of the function $K(u)$ lie only in the real regions.

Theorem. If the residues of the function $K(u)$ at the positive poles are of the same sign as the imaginary part, the integral equation (1.1) cannot have more than one solution in the space $L_\alpha$, $\alpha > 1$.

Proof. Assuming that $q(r) \in L_\alpha$, $\alpha > 1$, we conclude that

$$Q(u) \equiv \int_0^a q(\rho) J_0(u\rho)\rho\, d\rho \in L_\beta, \quad 1 < \beta < 2 \tag{2.1}$$

Let us multiply the homogeneous equation (1.1) by the function $\bar{q}(r)$ (the complex-conjugate), and let us integrate on the half-axis $[0, \infty]$. We then deform the contour $\Gamma$ onto the real axis. Taking account of the simplicity of the poles and the regularity of the function $Q(u)$, we arrive at the relationship

$$i \sum_{k=0}^{n} \int_{B_{2k}}^{B_{2k+1}} \operatorname{Im} K(u) |Q(u)|^2 du + \tag{2.2}$$

$$vp \int_0^\infty \operatorname{Re} K(u) |Q(u)|^2 du + i\pi \sum_{r=1}^{N} \{[K^{-1}(\zeta_r)]'\}^{-1} |Q(\zeta_r^2)|^2 \equiv 0$$

($\zeta_r$ are real positive poles of the function $K(u)$, $r = 1, 2, \ldots, N$). By separating real and imaginary parts in the last equation and taking account of the conditions of the theorem, we conclude that

$$Q(u) \equiv 0, \quad u \in [B_{2s}, B_{2s+1}]$$

By virtue of (2.1) the function $Q(u)$ is entire, hence $Q(u) \equiv 0$. It hence follows that $q(r) \equiv 0$. The condition $q(r) \in L_\alpha$, $\alpha > 1$ is used to give a foundation to the convergence of the integral in the relationship (2.2). The theorem is proved.

To give a foundation to the solvability of the integral equation (1.1), we reduce it to a Fredholm equation of the second kind. Several methods exist for such a reduction [2, 3]. The Fredholm equation which is also used to construct the approximate solution is presented below

$$y(z) = \frac{1}{4\pi^2} \int_{\Gamma_2} \int_{\Gamma_1} \frac{K_+(\alpha) C(\alpha, u) y(u)\, du\, d\alpha}{(\alpha - z)(\alpha^2 - u^2) K_+(u)} + F(z) \tag{2.3}$$

$$F(z) = \frac{1}{4\pi^2} \int_{\Gamma_2} \frac{D(\alpha) K_+(\alpha)}{(z - \alpha)} \varkappa_1(\alpha)\, d\alpha, \quad \varkappa_1(\alpha) = [i\sqrt{a} H_0^{(2)}(\alpha a)]^{-1}$$

$$q(r) = \int_0^\infty J_0(\xi r) K^{-1}(\xi) F(\xi) \xi\, d\xi + \int_\sigma J_0(ur) K_+^{-1}(u) y(u) \varkappa_2(u)\, du \tag{2.4}$$

$$C(\alpha, u) = -\varkappa_1(\alpha)\varkappa_2(u) G(\alpha, u) - (\alpha + u), \quad \varkappa_2(u) = \pi\sqrt{a}\, u H_0^{(2)}(ua)$$

$$G(\alpha, u) = a\alpha H_1^{(2)}(\alpha a) J_0(ua) - uaH_0^{(2)}(\alpha a) J_1(ua)$$

$$D(\alpha) = \int_0^\infty \frac{G(\alpha, \xi) F(\xi)}{(\xi^2 - \alpha^2) K(\xi)} \xi\, d\xi, \quad f(r) = \int_0^\infty F(\xi) J_0(\xi r) \xi\, d\xi$$

The contours $\Gamma_1$ and $\Gamma_2$ lie in the domain of regularity of the function $K(u)$ and their ends approach infinity in the lower half-plane, where the contour $\Gamma_2$ lies above the contour $\Gamma_1$. The contour $\sigma$ consists of the contour $\Gamma$ and its symmetric image with respect to the origin, in the left half-plane.

Equation (2.3) reduces to a Fredholm equation of the second kind on the contour $\Gamma_2$ in the class of functions continuous with weight $z^\lambda$, $0 < \lambda < 1$.

We shall consider the function $f(r)$ to be twice continuously differentiable. We represent the operator in the right side of (2.3) in a form acting from the contour $\Gamma_1$ again to the contour $\Gamma_1$. To this end, we represent it as follows by continuing the double integral analytically in the lower half-plane:

$$y(z)\left[1 - \frac{C(z,z)}{2z}\right] = \frac{iK_+(z)}{2\pi}\int\limits_{\Gamma_3}\frac{C(z,u)\,y(u)\,du}{(z^2-u^2)K_+(u)} +$$
$$\frac{1}{4\pi^2}\int\limits_{\Gamma_4}\int\limits_{\Gamma_3}\frac{K_+(\alpha)\,C(\alpha,u)\,y(u)\,du\,d\alpha}{(\alpha-z)(\alpha^2-u^2)K_+(u)} + F(z)$$

Here $z$ lies below the contour $\Gamma_3$ while the contour $\Gamma_4$ lies above the contour $\Gamma_3$.

Let us select the contour $\Gamma_3$ in the domain of regularity of $K(u)$ on which

$$1 - C(z,z)\,/\,(2z) \neq 0$$

We express $y(z)$ from the relationship obtained and insert it into the relationship (2.3). We consequently obtain an equation for $y(z)$ on the contour $\Gamma_1$. It can be proved that the operator in the right side of this equation is completely continuous in the space of continuous functions on the contour $\Gamma_1$ with weight $z^\lambda$, $0 < \lambda < 1$.

Since (2.3) is a Fredholm equation and has a unique solution, as has been proved, then it is solvable. Therefore, for every $f(r) \in c_2(0, a)$ there is found a unique function $q(r) \in L_\alpha$, $\alpha > 1$ which converts (1.1) into an identity.

We now study the properties of the function $q(r)$. It follows from the representation (2.3) that

$$y(z) \sim c\,/\,|z|, \quad z \to \infty, \quad z \in \Gamma_2$$

$$y(u)\,\varkappa_2(u) \equiv z(u) \sim \frac{c}{\sqrt{u}}\,e^{-iua}$$

Then

$$q(r) \sim c\int\limits_a^\infty e^{-u(a-r)}\,u^{-1/2}\,du, \quad r \to a \tag{2.5}$$

$$q(r) \sim c\,(a-r)^{-1/2}, \quad r \to a, \ |r| < a$$

Therefore, $q(r)\sqrt{a^2 - r^2} \in c\,(0, a)$. Consequently, the initial integral equation is correctly solvable and the following correctness relationships hold

$$\|q\|_{c_b} = \|q\sqrt{a^2-r^2}\|_{c(0,\,a)} \leqslant \|f\|_{c_2}$$

Now, using the theorem of the theory of linear operators, we can approximate the function $K(u)$ by the function $K^*(u)$ proceeding from the condition

$$\|K^\circ - K^{\circ *}\|_{c_b \to c_2} < \varepsilon$$

Here $\varepsilon > 0$ is a sufficiently small number, and the operator should act from the space $c_b$ to the space $c_2$. This latter permits replacement of the function $K(u)$ by another, close to it.

The possibility of constructing an approximation to any degree of accuracy is achieved by using interpolation polynomials [4]. In particular, the approximate function $K^*(u)$ can be represented in the form

$$K^*(u) = c^2 \prod_{i=1}^{M} (u^2 - z_i^2)\left[\sqrt{u^2+B^2} \prod_{j=1}^{M} (u^2 - \zeta_j^2)\right]^{-1} \tag{2.6}$$

Consequently

$$K_{\pm}^*(u) = c \prod_{i=1}^{M} (u \pm z_i) \left[\sqrt{B \mp iu} \prod_{j=1}^{M} (u \pm \zeta_j)\right]^{-1} \tag{2.7}$$

The branch point $u = \pm iB$ is selected as a function of the required conditions of the problem (the magnitude of the parameter $a$). We select $B$ from the condition $Ba \gg 1$. Inserting $K_{\pm}^*(u)$ into (2.3) and deforming the contour of integration downward to the branch point, we obtain the representation

$$y(z) = \frac{1}{4\pi^2} \int_{\Gamma_1^\circ} \frac{y(u) A(u, z)}{K_+^*(u)} du + \frac{1}{2\pi i} \sum_{j=1}^{M} \frac{y(-z_j) A(-z_j, z)}{[K_+^*(-z_j)]'} + \frac{1}{4\pi^2} B(z) \tag{2.8}$$

where

$$A(u, z) = A^*(u, z) + \int_{\Gamma_2^\circ} \frac{K_+^*(\alpha) C(\alpha, u)}{(\alpha - z)(\alpha^2 - u^2)} d\alpha \tag{2.9}$$

$$B(z) = B^*(z) + \int_{\Gamma_2^\circ} \frac{D(\alpha) K_+^*(\alpha) \varkappa_1(\alpha)}{z - \alpha} d\alpha$$

$$A^*(u, z) = 2\pi i \sum_{i=1}^{M} \frac{C(-\zeta_i, u)}{(\zeta_i + z)(\zeta_i^2 - u^2)[\{K_+^*(-\zeta_i)\}^{-1}]'}$$

$$B^*(z) = -2\pi i \sum_{l=1}^{M} \frac{D(-\zeta_l) \varkappa_1(-\zeta_l)}{(z + \zeta_l)[\{K_+^*(-\zeta_l)\}^{-1}]'}$$

The contours $\Gamma_i^\circ$ are obtained here as a result of deforming the contours $\Gamma_i$ in the lower half-plane. It can be shown that for sufficiently small $B$ the integral terms are small and can be discarded. Discarding the integral terms in (2.8) and (2.9) and setting $z = -z_m$ in (2.8), we obtain a linear algebraic system to determine $y(-z_m)$, $m = 1, 2, \ldots, M$

$$y(-z_m) = \frac{1}{2\pi i} \sum_{j=1}^{M} \frac{y(-z_j) A^*(-z_j, -z_m)}{[K_+^*(-z_j)]'} + \frac{1}{4\pi^2} B^*(-z_m) \tag{2.10}$$

Having determined the values of $y(-z_m)$ from (2.10), we can write an approximate expression for $y(z)$

$$y(z) \cong \frac{1}{2\pi i} \sum_{j=1}^{M} \frac{y(-z_j) A^*(-z_j, z)}{[K_+^*(-z_j)]'} + \frac{1}{4\pi^2} B^*(z) \tag{2.11}$$

We obtain the solution of the initial problem by substituting the approximate value of $y(z)$ from (2.11) into (2.4). Without limiting the generality, we can set in (1.1)

$$f(r) = f J_0(\eta r), \quad f = \text{const}, \quad \eta = \text{const}$$

Then the approximate solution of (1.1) can be written as

$$q(r) = f \left\{ \frac{J_0(\eta r)}{K^*(\eta)} + 2\pi^2 i \sqrt{a} \sum_{m=1}^{M} \frac{J_0(-z_m r) z_m H_0^{(2)}(-z_m a)}{[K_+^*(-z_m)]'} y(-z_m) \right\} + O(\Gamma^\circ) \tag{2.12}$$

The formula obtained for $q(r)$ is effective for internal points of the domain. The behavior as $r \to a$ is described by (2.5).

3. In the problem under consideration, the function $K(u)$ satisfies the properties listed in Sect. 2. For fixed $\varkappa_2$ we hence have a real interval $[A_1, A_2] = [\theta_2', \infty]$ and a complex interval $[B_1, B_2] = [0, \theta_2']$. The signs of the imaginary component and the residues at the poles, which all turn out to be negative in the example presented below, are verified without difficulty in a numerical investigation of the real and imaginary parts of the function $K(u)$

Let us present the sequence of operations needed to compute the state of stress and strain by the method of this paper.

1). Having constructed the integral equation (1.1), we approximate the function $K(u)$ in (1.2) by an expression of the form (2.6) on the axis $[0, \infty]$ (the real zeros and poles of $K(u)$ are evaluated first). Approximating polynomials of different kinds can be used here [3, 4].

2). On the basis of the approximation introduced, we solve the system (2.10). We consequently obtain the distribution of the contact stresses in (2.12).

At this time the algorithm presented has been realized by a program permitting execution of the computations for any relationships between the parameters of the problem.

As an illustration the following case is considered:

$$\varepsilon^2 = 1.25, \quad \eta = 0, \quad E_1 = 1.2\cdot 10^6, \quad E_2 = 1.5\cdot 10^6, \; \nu_1 = 0.27, \; \nu_2 = 0.26, \; \varkappa_2 = 1.1.$$

The coefficients of the approximations (2.6), (2.7) have the values ($M = 7$):

$$z_1 = 1.760366, \quad \zeta_1 = 2.092845, \quad z_2 = 1.82099 + 0.63240i, \; z_3 = 0.518977 + 0.578446i,$$
$$z_4 = -2.57811 + 1.65149i, \; z_5 = -1.58223 + 0.260871i, \; z_6 = -0.752302 + 0.608482i,$$
$$z_7 = 2.99651 + 4.76229i.$$

The quantities $\zeta_i$, $i = 2, 3, \ldots, 7$ are selected from all the values

$$5000^{1/12}(-8{,}89814 + 0.57663i)^{-1/12}$$

according to the condition $\operatorname{Im} \zeta_i > 0$. The error in the approximation presented does not exceed 10% for small $|u|$, and is practically zero for $|u| > 10$. Graphs of the real and imaginary components of the amplitude function $q(r)$ are presented in Fig. 1 for $a = 1$. The real part of $q(r)$ is superposed by the solid line, and the imaginary part by the dashes. The expression

$$\sigma_z(r) = \operatorname{Re}[q(r)\, e^{-i\omega t}], \qquad r < a$$

yields the magnitude of the stresses under the stamp.

The displacement of points of the layer surface outside the stamp can be obtained from (1.1) for known $q(r)$ by evaluating the integral for $a < r$. In the case $f(r) = fJ_0(\eta r)$ the approximate value of the amplitude function of the displacements can be written in the form

$$W(r) \cong \frac{if}{8} \sum_{n=1}^{M} \Big[ \frac{-a\zeta_n J_0(\eta a) J_1(-\zeta_n a) - \eta a J_1(a\eta) J_0(-\zeta_n a)}{(\zeta_n^2 - \eta^2) K(\eta)} +$$
$$2\pi^2 i \sqrt{a} \sum_{m=1}^{M} \frac{z_m H_0^{(2)}(-z_m a)\, y(-z_m)}{[K_+^*(-z_m)]'} \times$$

$$\left.\frac{-\zeta_n a J_0(-z_m a) J_1(-\zeta_n a) + z_m a J_1(-z_m a) J_0(-\zeta_n a)}{(\zeta_n^2 - z_m^2)}\right] \times \frac{\zeta_n H_0^{(2)}(-\zeta_n r)}{\{[K^*(-\zeta_n)]^{-1}\}'}, \quad r \gg a$$

The expression

$$w(r) = \mathrm{Re}\,[W(r)\, e^{-i\omega t}]$$

yields the magnitude of the displacements of the layer surface outside the stamp.

It should be noted that the formulas presented for the amplitude function of the contact stresses and the layer surface displacements outside the stamp are written down under the assumption that all the complex zeros and poles of the function $K^*(u)$ are simple. This is realized automatically in the construction of the function $K^*(u)$ in the form (2.6).

The problems examined may be used as a model in constructing the initial data for an investigation of the transmission of vibrations between foundations, as well as the simplest model for a vibration investigation of soil by means of data on wave field excited on the surface. Namely, by having a sufficient set of solutions of the problem for different elastic and geometric values of the base, the parameters of the base can be predicted from the condition of best agreement between experimental and theoretical results by comparing the amplitude and phase values of the wave fields.

We note that the problem considered for an ideally elastic foundation is more complex than the corresponding problem for a viscoelastic medium. The method proposed is valid for this latter case as well.

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